Journal of Engineering Mathematics, Vol. 7, No. 3, July 1973
Noordhoff International Publishing–Leyden
Printed in The Netherlands

On the optimal estimation of the lateral position of a ship using yaw-angle measurements

R. P. VAN DER VET

Delft University of Technology and Koninklijk Instituut voor de Marine, Den Helder, the Netherlands★

(Received November 1, 1972)

SUMMARY
The Kalman filter theory has been used to derive a set of difference equations by which the lateral position of a ship relative to the desired (straight) course can be estimated from measured yaw-angle values containing noise. Special attention has been paid to the standard deviation of the estimation error.

1. Introduction

The position of a ship following a straight course is determined at intervals to make certain that the course is maintained. For this purpose several different types of electronic equipment are available, such as the Decca navigation system. The helmsman is instructed between two position determinations to keep the ship at the prescribed yaw angle. This, however, is no guarantee that the ship is steaming along the desired track. For, she is subjected to several, partly stochastic, disturbances, such as wind, current and waves, and the lateral movement of the ship between two position determinations thus is not known exactly.

In this paper an optimal estimate of the lateral position of a ship following a prescribed straight course is made on the basis of disturbed yaw-angle measurements. The estimate is obtained by application of the Kalman filter theory [3]. In the present problem the disturbance vector is multiplied by an input matrix, which gives a slight modification of the filter derived in [3].

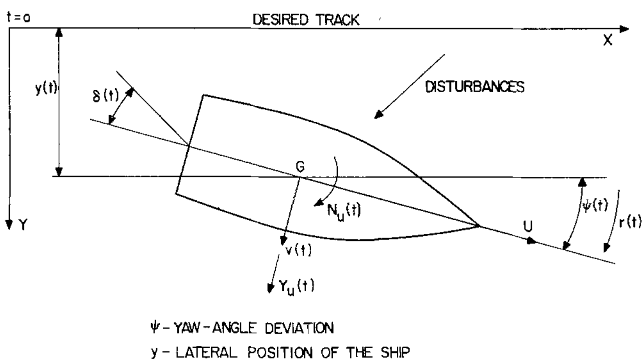

Figure 1.

It is supposed that at the moment $t=0$ the lateral position y of the ship relative to the desired track is normally distributed, with known mean and variance (Figure 1). From that moment, the ship steaming at the prescribed yaw angle, the estimate of the lateral position is made. Attention will be paid particularly to the standard deviation of the estimation error. Pitching, rolling and heaving of the ship are neglected.

2. Description of the system

The dynamical behaviour of a ship in the horizontal plane can be described sufficiently exact

★ Present address: Koninklijke/Shell-Laboratorium, Badhuisweg 3, Amsterdam-N., the Netherlands.

by a simultaneous system of linear differential equations if the deviations from the equilibrium state are small.

This system is represented, according to [4], by:

$$\begin{aligned}(M-Y_{\dot v})\dot v - Y_v v - Y_{\dot r}\dot r + (MU - Y_r) r &= Y_\delta \delta + Y_u \\ -N_{\dot v}\dot v - N_v v + (I_z - N_{\dot r})\dot r - N_r r &= N_\delta \delta + N_u\,,\end{aligned} \tag{2.1}$$

where $v(t)$ is the transverse velocity of the ship (see also Figure 1), $r(t)$ the yaw angular velocity and $\delta(t)$ the rudder angle. $Y_u(t)$ and $N_u(t)$ are, respectively, the external force and the moment acting on the ship. A dot above a symbol denotes its derivative with respect to time. The coefficients Y_v, N_v, Y_δ, etc. of the system are dependent on the ship considered and on her forward speed.

The equations can be written in a matrix notation:

$$\begin{bmatrix}\dot v\\ \dot r\end{bmatrix} = B\cdot\begin{bmatrix}Y_v & MU - Y_r\\ N_v & N_r\end{bmatrix}\begin{bmatrix}v\\ r\end{bmatrix} + B\cdot\begin{bmatrix}Y_\delta\\ N_\delta\end{bmatrix}\delta + B\cdot\begin{bmatrix}Y_u\\ N_u\end{bmatrix} \tag{2.2}$$

with

$$B = \begin{bmatrix}M - Y_{\dot v} & -Y_{\dot r}\\ -N_{\dot v} & I_z - N_{\dot r}\end{bmatrix}^{-1}.$$

A first-order approximation of the velocity of the ship in the Y-direction (Figure 1) is

$$\dot y = v + U\psi \tag{2.3}$$

Finally, the yaw angle ψ satisfies

$$\dot\psi = r \tag{2.4}$$

By combining (2.2), (2.3) and (2.4), the following state equation is obtained:

$$\dot x(t) = \Phi^* x(t) + \Delta^* \delta(t) + \Theta^* u(t)\,, \tag{2.5}$$

where $x'(t) = (v(t), r(t), \psi(t), y(t))$ is the state vector*, $u'(t) = (Y_u(t), N_u(t))$ is the vector of the external disturbances, Φ^* is a 4×4 system matrix, Δ^* a 4×1 input vector and Θ^* a 4×2 input matrix.

The matrices of the state equation contain the coefficients and the forward speed of the ship and can easily be determined if the ship and her forward speed have been chosen (see section 6).

To make the system suitable for digital computation it is put in a discrete form by a well-known technique [5]. With a sampling period of T seconds the following state equation in discrete form is obtained:

$$x(k+1) = \Phi x(k) + \Delta\delta(k) + \Theta u(k)\,, \tag{2.6}$$

where

$$\Phi = e^{\Phi^* T}\;;\quad \Delta = \left\{\int_0^T e^{\Phi^*\tau} d\tau\right\}\Delta^*\;;\quad \Theta = \left\{\int_0^T e^{\Phi^*\tau} d\tau\right\}\Theta^* \tag{2.7}$$

and where k is the time index.

As mentioned in the introduction, the estimate of the lateral position of the ship is made on the basis of yaw-angle measurements containing noise.

Thus for the observation scalar $z(t)$ is written:

$$z(t) = \psi(t) + w(t) = c'\cdot x(t) + w(t)\,, \tag{2.8}$$

where $c' = (0, 0, 1, 0)$ and $w(t)$ is the measuring noise scalar. This equation can easily be reduced to the discrete form

$$z(k) = c' x(k) + w(k)\,. \tag{2.9}$$

* The transpose of a vector or matrix will be denoted by a prime.

3. Introduction to the estimation problem

The input to system (2.6) consists of the rudder angle $\delta(k)$ and the disturbance vector $u(k)$. The latter contains both deterministic disturbances (wind, current) and stochastic disturbances (waves).

As the system is linear, its response can be obtained by separately adding up the responses to the deterministic disturbances (including the rudder angle $\delta(k)$) and the responses to stochastic disturbances. From now on only the stochastic disturbances, due to the waves, will be considered. This means that $u(k)$ now is a stochastic vector. The probability distribution of this vector will be specified later.

Thus, as starting point of the system we now have:

$$x(k+1) = \Phi x(k) + \Theta u(k)\,, \tag{3.1}$$

the observations being described by (2.9):

$$z(k) = c' x(k) + w(k)\,.$$

With respect to $u(k)$ and $w(k)$, the following assumptions are made:

- $u(k)$ and $w(k)$ are mutually independent and normally distributed. Both are independent from one time instant to the other and independent of the initial state $x(0)$. The mean and variance are:

$$\begin{aligned} &E[u(k)] = 0\;; \quad E[w(k)] = 0\;;\\ &E[u(k)\,,\; u(k)'] = Q\;;\\ &E[w(k)^2] = q\;; \end{aligned} \tag{3.2}$$

  where Q is a positive definite 2×2 diagonal matrix and q a positive scalar.
- The initial state $x(0)$ is a normally distributed stochastic vector with

$$E[x(0)] = 0\;; \quad E[x(0)\,,\; x(0)'] = Q_0\,, \tag{3.3}$$

  where Q_0 is a positive definite 4×4 diagonal matrix.

The estimation problem can now be formulated as follows: given the observed values $z(0)\ldots z(k)$, find an estimate $\hat{x}(k)$ of $x(k)$ which minimizes some performance function. An obvious way of choosing $\hat{x}(k)$ is to require that the value chosen should minimize the mean square of the estimation error $\tilde{x}(k) = x(k) - \hat{x}(k)$. The optimal estimate of $x(k)$ is then obtained from:

$$\min_{\hat{x}(k)} E\{[x(k) - \hat{x}(k)]'[x(k) - \hat{x}(k)]\}\,, \tag{3.4}$$

given a set of observed values $z(0)\ldots z(k)$. From this follows the estimated value of $x(k)$ [2]:

$$\hat{x}(k) = E\{x(k) | z(0)\ldots z(k)\}\,. \tag{3.5}$$

For ease of reference, hereafter the symbols $\hat{x}(k|j)$ and $\tilde{x}(k|j)$ will be used to represent, respectively, the estimate and the estimation error of $x(k)$, given the observations $z(i)$ up to and including the time instant $j \leqq k$.

4. Estimation of the initial state

The estimated initial state follows from (3.5):

$$\hat{x}(0) = E[x(0) | z(0)]\,.$$

As $x(0)$ and $z(0)$ are normally distributed, this expectation can be separated into [1]:

$$E[x(0) | z(0)] = E[x(0)] + E[x(0)\,;\; z(0)'] \, E[z(0)^2]^{-1} [z(0) - E[z(0)]]\,. \tag{4.1}$$

Since $E[x(0)] = 0$ and $E[w(0)] = 0$ and hence $E[z(0)] = 0$, this form can be reduced to:

$$\hat{x}(0)=E[x(0)|z(0)]=K(0)z(0)\,, \tag{4.2}$$

where

$$K(0)=Q_0c[c'Q_0c+q]^{-1}\,, \tag{4.3}$$

being a 4×1 vector obtained by applying the equations (2.9), (3.2) and (3.3) to (4.1).

The estimation error is given by:

$$\begin{aligned}\tilde{x}(0)&=x(0)-E[x(0)|z(0)]\\&=[I-K(0)c']x(0)-K(0)w(0)\,,\end{aligned}$$

where I is the 4×4 identity matrix. Hence the covariance matrix of the estimation error satisfies:

$$P(0)=E[\tilde{x}(0)\,,\ \tilde{x}(0)']=[I-K(0)c']Q_0[I-K(0)c']'+qK(0)K(0)'\,,$$

which may be reduced to

$$P(0)=[I-K(0)c']Q_0\,. \tag{4.4}$$

5. The estimation of $x(k)$

In this section it will be shown that the best estimate $\hat{x}(k|k)$ of $x(k)$ can be obtained from the estimate $\hat{x}(k-1|k-1)$ and the observed value $z(k)$. Only the main features will be treated. For a more detailed treatment of the concepts used the reader is referred to [2]. Suppose that the values $z(0)\dots z(k-1)$ are known and that $\hat{x}(k-1|k-1)$ is determined. It is possible to orthogonalize the sequence $z(0)\dots z(k-1)$, that is to say there exists an orthonormal sequence $y(0)\dots y(j)$, $j\leq k-1$, with the property that each $y(i)$ is a linear combination of $z(0)\dots z(k-1)$ and *vice versa*. Therefore we can consider the linear space $Z(k-1)$ to consist of all linear combinations

$$\sum_{i=0}^{k-1} a_i z(i)\,.$$

The new observed value $z(k)$, which does not necessarily belong to $Z(k-1)$, can be uniquely decomposed into two parts, as follows: $z(k)=\hat{z}(k|k-1)+\tilde{z}(k|k-1)$ with $\hat{z}(k|k-1)\in Z(k-1)$ and $\tilde{z}(k|k-1)$ orthogonal to $Z(k-1)$. In fact $\hat{z}(k|k-1)$ can be considered to be the best estimate of $z(k)$, given $z(0)\dots z(k-1)$, i.e. $\hat{z}(k|k-1)=E[z(k)|z(0)\dots z(k-1)]$. The linear space $Z(k)$ is thus generated by $Z(k-1)$ and $\tilde{z}(k|k-1)$.

It follows from the above that, with substitution of the system equations, (3.1), the estimate of $x(k)$ can be written as

$$\begin{aligned}\hat{x}(k|k)=E[x(k)|Z(k)]&=E[x(k)|Z(k-1)]+E[x(k)|\tilde{z}(k|k-1)]\\&=\Phi\hat{x}(k-1|k-1)+E[x(k)|\tilde{z}(k|k-1)]\,,\end{aligned} \tag{5.1}$$

considering that $u(k-1)$ is independent of $z(0)\dots z(k-1)$, so that $E[u(k-1)|Z(k-1)]=0$. In the same way as in the estimation of the initial state (see eq. (4.1)) the term $E[x(k)|\tilde{z}(k|k-1)]$ is separated into

$$\begin{aligned}E[x(k)|\tilde{z}(k|k-1)]=E[x(k)]+E[x(k),\ \tilde{z}(k|k-1)']\,\cdot\\ \cdot\,E[\tilde{z}(k|k-1)^2]^{-1}\cdot[\tilde{z}(k|k-1)-E[\tilde{z}(k|k-1)]]\,.\end{aligned} \tag{5.2}$$

Application of (2.9), (3.1) and (3.2) yields the following relations:

$$E[x(k),\ \tilde{z}(k|k-1)']=[\Phi P(k-1)\Phi'+\Theta Q\Theta']c \tag{5.3}$$

$$E[\tilde{z}(k|k-1)^2]^{-1}=[c'\Phi P(k-1)\Phi'c+c'\Theta Q\Theta'c+q]^{-1} \tag{5.4}$$

$$E[\tilde{z}(k|k-1)]=0\,, \tag{5.5}$$

where

$$P(k)=E[\tilde{x}(k|k),\ \tilde{x}(k|k)']$$

being the 4×4 covariance matrix of the estimation error, for which a more useful equation will be derived later.

As in this case $E[x(k)]=0$, by substitution of (5.2) into (5.1) and application of (5.3), (5.4) and (5.5) we find that:

$$\hat{x}(k|k)=\Phi\hat{x}(k-1|k-1)+K(k)\tilde{z}(k|k-1)\,, \tag{5.6}$$

where

$$K(k)=S(k)c[c'S(k)c+q]^{-1} \tag{5.7}$$

$$S(k)\ =[\Phi P(k-1)\Phi'+\Theta Q\Theta']\,, \tag{5.8}$$

$K(k)$ being a 4×1 vector and $S(k)$ a 4×4 matrix.

From

$$\begin{aligned}\tilde{z}(k|k-1)&=z(k)-E[z(k)|z(0)\ \dots\ z(k-1)]\\&=z(k)-c'\Phi\hat{x}(k-1|k-1)\end{aligned}$$

we can conclude that

$$\begin{aligned}\hat{x}(k|k)&=\Phi\hat{x}(k-1|k-1)+K(k)[z(k)-c'\Phi\hat{x}(k-1|k-1)]\\&=A(k)\hat{x}(k-1|k-1)+K(k)z(k)\,,\end{aligned} \tag{5.9}$$

where

$$A(k)=[I-K(k)c']\,\Phi \tag{5.10}$$

being a 4×4 matrix.

From (5.9) it follows that for the estimation of $x(k)$ all the previous observations $z(0)\dots z(k-1)$ can be summarized by $\hat{x}(k-1|k-1)$. The term $\Phi\hat{x}(k-1|k-1)$ can be interpreted as the *a priori* estimate of $x(k)$ based on $\hat{x}(k-1|k-1)$. The second term in the first equation of (5.9) is the correction to the *a priori* estimate due to the actual value of the observation $z(k)$. The matrix $K(k)$ is weighting the difference between the observed value $z(k)$ and its *a priori* estimate.

It should be noted, however, that equation (5.9) yields for the system (3.1), the non-stochastic inputs not included. If these are involved the estimate will have the following form:

$$\hat{x}(k|k)=[I-K(k)c'][\Phi\hat{x}(k-1|k-1)+\Delta\delta(k-1)+\Gamma\gamma(k-1)]+K(k)z(k)\,,$$

where $\delta(k-1)$ is the rudder angle and $\gamma(k-1)$ the non-stochastic disturbances (wind and current) acting on the ship.

There remains to be derived a more useful equation for the covariance matrix $P(k)$ of the estimation error. Substitution of (5.9) and (5.10) into $\tilde{x}(k|k)=x(k)-\hat{x}(k|k)$ results in a recursive equation for the estimation error:

$$\tilde{x}(k|k)=[I-K(k)c'][\Phi\tilde{x}(k-1|k-1)+\Theta u(k-1)]-K(k)w(k)\,. \tag{5.11}$$

Thus there is obtained

$$\begin{aligned}P(k)&=E[\tilde{x}(k|k)\,,\ \tilde{x}(k|k)']\\&=[I-K(k)c']S(k)[I-K(k)c']'+q\cdot K(k)K(k)'\,,\end{aligned} \tag{5.12}$$

which can be reduced to

$$P(k)=[I-K(k)c']\,S(k)\,. \tag{5.13}$$

6. Calculations

The calculations can be made after specification of the covariance matrices in (3.2) and (3.3) and the coefficients of the ship (2.1). The ship considered here is one of the Todd Sixty Series. The dimensions are:

Length between perpendiculars	168.- m
Breadth	24.- m
Draught	9.59 m
Volume of displacement	26 970 m^3

The forward speed is 8.- m/s, which corresponds to a value of 0.20 of the Froude number. Using these data, the coefficients of (2.1) can be determined from [4]. Unfortunately, these coefficients are dependent on the frequency of the forces and moments acting on the ship. (For some examples see [4].) Therefore as nominal values we use the values at very low frequencies. To include the effect of higher frequencies, the nominal values are varied. Some additional remarks on this procedure will be made further on this section.

For all the calculations a sampling period $T=5$ seconds is chosen. Now it is possible to obtain the elements of the matrices Φ and Θ of (3.1). They will not be specified here.

Next the covariance matrices Q and Q_0 have to be specified. The diagonal elements of Q represent the variance of the force and of the moment exerted on the ship by the waves. The values of these variances have been obtained from [6]; they hold for some sea state. Averaged over the sampling period they are:

variance of the force: 10^{10} $(\text{kg})^2$

variance of the moment: $64 \cdot 10^{12}$ $(\text{kgm})^2$.

The diagonal elements Q_0 indicate the uncertainty about the elements of the initial state vector. We shall use the notation qo_{11}, qo_{22}, qo_{33} and qo_{44} for the variances of $v(0)$, $r(0)$, $\psi(0)$ and $y(0)$, respectively. To express the uncertainty in respect of the values $v(0)$ and $r(0)$ (which are not measured) rather large values are chosen for qo_{11} and qo_{22}, namely 100 m^2/s^2 for qo_{11} and 25 $1/\text{s}^2$ for qo_{22}. For the other elements, which are measured or specified at $t=0$, the following values are chosen:

$$qo_{33} = 0.0025 \quad \text{and} \quad qo_{44} = 10\,000 \text{ m}^2 .$$

The values chosen so far will be kept constant, which means that the estimate is made for a particular ship in a particular sea state and with specific assumptions about the initial state. In order to investigate how the estimate is affected by the variation of the coefficients of the ship in relation to the nominal values and by the variance q of the noise contained in the observation, these are chosen so as to have different values, namely:

TABLE 1

	q	R	S		q	R	S
1	0.0005	0	0	4	0.0005	0	+0.2
2	0.00005	0	0	5	0.0005	0	−0.2
3	0.000005	0	0	6	0.0005	+0.2	−0.2

The cases 1, 2 and 3 represent the effect on the estimate of the noise contained in the observation. The cases 4, 5 and 6 include the effect of a variation of the coefficients of the ship. These coefficients are divided in two groups, namely group R $(M-Y_{\dot{v}},\ Y_{\dot{r}},\ N_{\dot{v}},\ I_z-N_{\dot{r}})$ and group S $(Y_v,\ N_r,\ MU-Y_r,\ N_v)$. The values in the respective column are the factors by which the coefficients are varied. This division is made because the dynamic stability of the ship is determined mainly by the coefficients in group S. More exactly, the ship may be considered to be more stable in case 4 than in case 5. For each of these cases the relevant quantities, i.e. the matrices $A(k)$, $K(k)$ and $P(k)$, can be calculated with the aid of equations (5.8), (5.7), (5.10) and (5.13) in that order. The first step is taken with (4.3) and (4.4).

The estimate of the lateral position of the ship relative to the desired course is now obtained from (5.9):

$$\hat{y}(k+1) = A_{41}(k)\hat{v}(k) + A_{42}(k)\hat{r}(k) + A_{43}(k)\hat{\psi}(k) + A_{44}(k)\hat{y}(k) + K_4(k)z(k), \tag{6.1}$$

where $A_{4i}(k)$; $i = 1(1)4$ are elements of the matrix $A(k)$ and $K_4(k)$ is an element of the 4×1 vector $K(k)$.

The standard deviation of the estimation error can easily be determined from (5.13):

$$s_y(k) = \{E[\tilde{y}(k)^2]\}^{\frac{1}{2}} = \{P_{44}(k)\}^{\frac{1}{2}}. \tag{6.2}$$

It is assumed that the estimate covers a period of 900 seconds.

The calculations were carried out on the X-8 digital computer of "Reactor Centrum Petten" in the Netherlands.

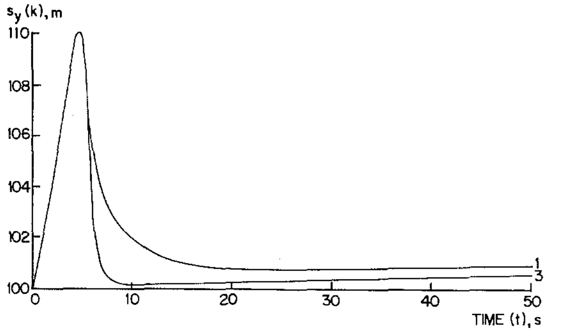

Figure 2. Transient response of the standard deviation $s_y(k)$.

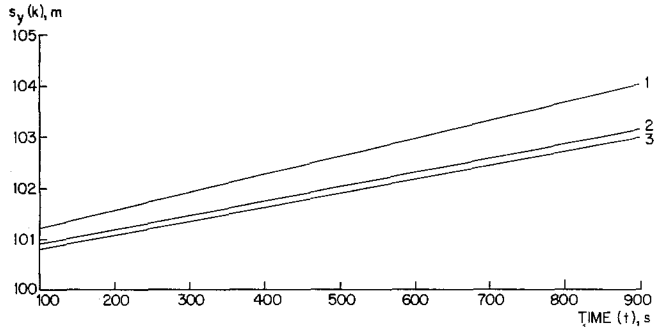

Figure 3. The standard deviation $s_y(k)$.

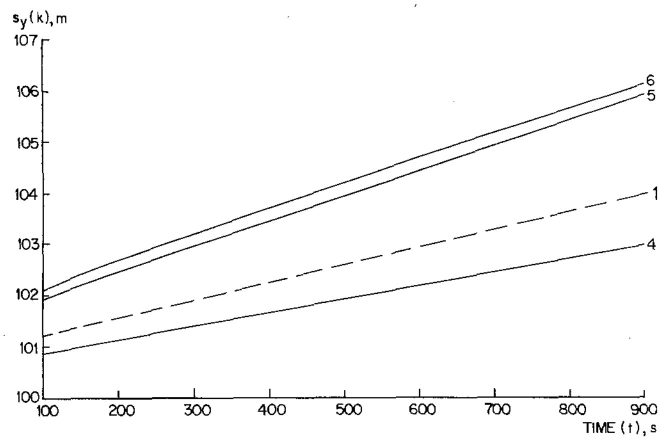

Figure 4. The standard deviation $s_y(k)$.

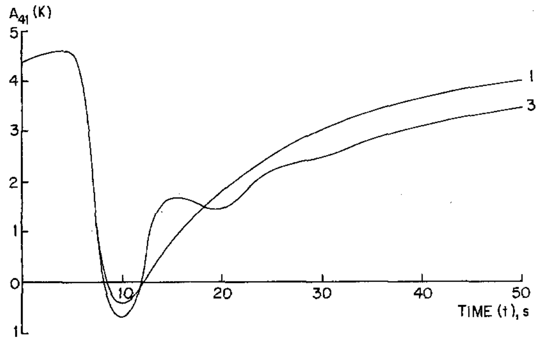

Figure 5. Transient response of the coefficient $A_{41}(k)$.

7. Results

As expected, the standard deviation $s_y(k)$ increases with time (Figures 3 and 4); the lateral movement of the ship not being observed, the uncertainty of the lateral position increases. Both the variance q of the noise contained in the observation and the values of the coefficients of the ship affect the course of the standard deviation; see Figures 3 and 4 for the results. The smaller the permitted variance q, the better the course of the standard deviation $s_y(k)$. However, the yaw angle measurement sensors then become more expensive. The influence of the coefficients in group R (case 6 in Table 1) is hardly significant, as became apparent also from other calculations not reported here.

We might conclude from Figure 4 that the estimate is better according as the dynamical stability of the ship is higher (compare cases 4 and 5). This, however, is not true in general. Other calculations showed that even for a course-unstable ship a good estimate can be made. It was also found that larger changes in the coefficients in groups R and S do not greatly affect the estimate; all results were better than those for case 5 represented in Figure 4. This case may be considered an upper limit.

The standard deviation has a transient response (Figure 2). This follows from (6.1) with $k=0$ and is due to the great variance, qo_{11} and qo_{22}, accepted for $v(0)$ and $r(0)$ respectively. The estimates for $v(k)$ and $r(k)$ become better when $k \geqq 1$, as a result of which $s_y(k)$ decreases until the steady state is attained.

The coefficients A_{41}, A_{42}, A_{43} and K_4 of the estimator (6.1) also have a transient response (see Figure 5 for A_{41}), as follows from an analysis of (5.10), (5.7) and (5.8). The stationary values are presented in the table below. The numbers in brackets indicate the time in seconds at which the values became stationary. A_{44} has the constant value 1.

TABLE 2

	A_{41}	A_{42}	A_{43}	K_4		A_{41}	A_{42}	A_{43}	K_4
1	+4.411 (200)	+50.570 (285)	+37.413 (280)	+2.587 (280)	4	+4.324 (140)	+42.516 (240)	+36.679 (210)	+3.321 (210)
2	+4.217 (185)	+111.374 (280)	+53.750 (265)	−13.750 (265)	5	+4.499 (230)	+61.983 (375)	+38.794 (315)	+1.206 (315)
3	+4.052 (175)	+163.059 (295)	+67.637 (260)	−27.637 (260)	6	+4.584 (250)	+61.351 (420)	+37.531 (380)	+2.469 (380)

There is some agreement between these stationary values and the coefficients of the differential equation (2.3). The discrete version of this equation with a sampling period $T=5$ seconds and a ship speed $U=8.$- m/s is:

$$y(k) = 5v(k-1) + 40\psi(k-1) + y(k-1)\,. \tag{7.1}$$

A_{41} obtains a stationary value of approximately 5. The value of A_{44} is the same as in (7.1). The sum of A_{43} and K_4 always amounts to 40. In the estimator (6.1) this amount is obviously divided over the yaw-angle observation and estimation. The more noise is contained in the observation, the greater the value attached to the estimate of the yaw angle.

8. Conclusion

The forces and moments acting on the ship due to the waves have been approximated by a white noise process. Starting from this approximation a very good estimate of the lateral position of the ship can be made. This estimate is better according as less noise is contained in the yaw-angle measurement. However, the sensors will then be more expensive. Upon a change in the system coefficients of the ship the standard deviation of the estimation error remains within reasonable bounds.

Acknowledgement

The author is indebted to the Koninklijk Instituut van de Marine, Den Helder, where the greater part of the investigation was performed. His thanks are due particularly to Dr. Ir. J. K. Zuidweg for the many fruitful discussions on the subject and to Professor Dr. R. Timman for his willingness to accept this study as a part of the work required to take his degree in mathematical engineering.

REFERENCES

[1] R. Deutsch, *Estimation Theory*, Prentice Hall Inc., London (1965).
[2] J. L. Doob, *Stochastic Processes*, John Wiley, London (1953).
[3] R. E. Kalman, A new Approach to Linear Filtering and Prediction Problems, *J. Basic Eng.*, (1960) 35–45.
[4] G. van Leeuwen, *The Lateral Damping and added Mass of a Horizontally Oscillating Ship Model.* Netherlands Research Centre T.N.O. for Ship-building and Navigation, Report No. 65 S, December (1964).
[5] L. A. Zadeh and C. A. Desoer, *Linear System Theory*, McGraw-Hill, New York (1963).
[6] J. K. Zuidweg, Information from Laboratorium voor Reken- en Regeltechniek, Koninklijk Instituut voor de Marine, Den Helder, June (1969).